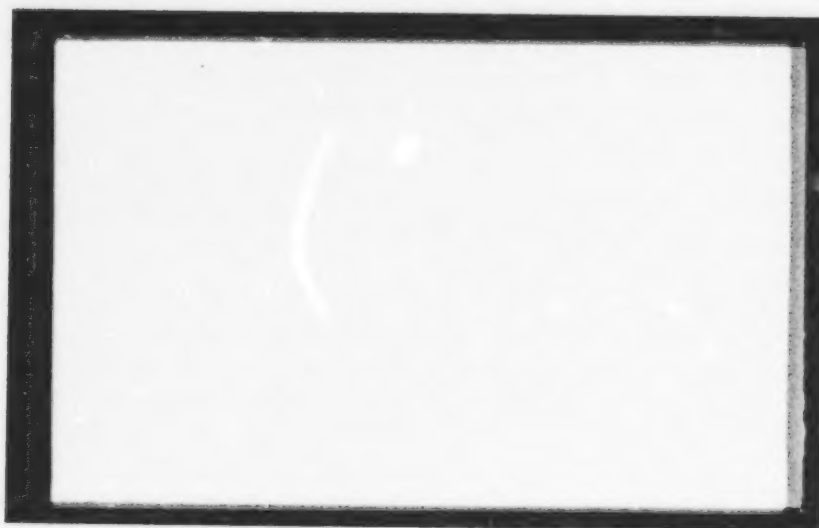


UNIVERSITÉ DE MONTRÉAL



CENTRE DE RECHERCHE SUR LES TRANSPORTS



CASE POSTALE 6126, STATION A, MONTRÉAL, QUÉBEC, H3C 3J7

EXPECTED COMPLETION TIME IN 'PERT' NETWORKS

by

Pierre Robillard

and

Michel Trahan

This research was supported by a grant from the Ford Motor Company of Canada.

Université de Montréal

Publication # 167 - Département d'informatique

Publication # 3 - Centre de Recherche sur les Transports

March 1974

ABSTRACT

This note is concerned with a technical improvement in PERT Critical Path Methodology: we introduce a general approach to obtain lower bounds for the expected completion time in PERT networks whose activity durations are random (discrete or continuous) variables. We show how some well known bounds can be derived as particular cases of this general approach which is based on the Jensen's inequality. We assume that the set of activity durations directed into a node is independent from the set of activity durations directed into another node and that all the activity durations have a finite positive range.

Key-words: PERT - CPM - lower bounds - stochastic networks - expected completion time - Jensen's inequality.

RESUME

Ce travail apporte des améliorations techniques à la méthodologie PERT ainsi qu'à celle du CPM: nous proposons une approche pour obtenir des bornes inférieures à la durée moyenne d'un programme représenté par un réseau PERT. Nous montrons comment des bornes connues tiennent lieu de cas particuliers dans cette approche généralisée qui est basée sur l'inégalité de Jensen. Nous supposons que l'ensemble des temps requis pour accomplir les activités associées aux arcs entrant dans un noeud est indépendant de l'ensemble des temps associés aux arcs entrant dans un autre noeud. De plus, nous supposons que les temps requis pour accomplir les activités sont des variables aléatoires discrètes ou continues définies sur un ensemble positif borné.

Mots-clés: PERT - CPM - bornes inférieures - réseau stochastique -
durée moyenne - inégalité de Jensen.

INTRODUCTION

During the past several years, new techniques based on network models have been developed to aid management in planning and controlling projects. The PERT technique has received widespread interest and has made great strides. But the fact remains that the evaluation of the expected completion time of a project still sets problems when the activity durations are random variables. The expected completion time can be obtained through the density function of the total completion time and the definition of the mathematical expectation. Unfortunately, the evaluation of this density function requires a great deal of work: Martin (1965). Thus, the research was directed towards an approximation of the expected completion time. In most cases, planning managers only need to have a rough idea of the expected completion time, in order that an approximation may be satisfactory. An approximation of the density function can be obtained by the Monte Carlo techniques: Burt and Garman (1971), Moder and Phillips (1970) and Pritsker and Kiviat (1969). Lindsey (1971) developed a method of calculating the expected completion time based on a model approximating the actual network. Bounds are also available: an upper bound is produced from the lower bound approximation of the distribution function of the completion time proposed by Robillard and Trahan (1974). The usual practice to obtain a lower bound (optimistic estimate) is simply to replace each random variable on arc by its expected value and to use the well known CPM: Kaufman and Desbazeille (1969). This lower bound has the inconveniency of not changing with the variance of the distribution function associated to each arc. In many situations this

CPM estimate may be very bad: MacCrimmon and Ryavec (1964). This lower bound can be improved by simulation techniques: Moder and Phillips (1970). Fulkerson (1962) proposed a method of calculating a lower bound better than the CPM bound when the activity durations are discrete random variables: this bound is function of the variance associated with each arc. Clingen (1964) generalized the Fulkerson's (1962) bound for the continuous cases. Elmaghraby (1967) improved this bound. In this note, we describe a general approach to obtain lower bounds to the expected completion time. We show that the CPM's and the Fulkerson's (1962) bounds, among others, can be derived directly from this approach. This approach allows the improvement of the existing bounds: a particular case is considered where the proposed bound is always better than the Fulkerson's (1962) bound and can overstep the Elmaghraby's (1967) estimate if we are willing to pay a little more. We assume that the activity durations discrete or continuous have a finite positive range and that the set of activity durations directed into a node is independent from the set of activity durations directed into another node. This paper is divided in four sections: in the first section (PERT networks), we outline the basis PERT technique and its graphical representation. In section two (Notation and definitions), the relevant notation and definitions are introduced. Section three (Lower bounds for the expected completion time) is concerned with lower bounds for the expected time required to complete all the activities of the project. Some considerations about the evaluation of a new bound are pointed out in section four (A new bound).

1. PERT networks

The technique known under the name of PERT is concerned with a set of activities converging to the realization of an objective. The times to perform each of the activities are known and also their relative ordering. An activity has a unique definable beginning and ending. The beginning and ending points of the activities are called events. A PERT network is a graphical representation of those activities: each activity is represented by an arc having a time variable to which it is associated. The nodes are the events. The beginning node of an arc corresponds to the initialization of the associated activity and the ending node to its completion. The position of an activity is dictated by the following rule: all the activities associated with the incoming arcs to a node must be terminated before those associated with the outgoing arcs at that node could start. In consequence, PERT technique leads to the PERT networks which are directed, acyclic networks.

2. Notation and definitions

Suppose given a PERT network G composed of n nodes and m arcs. Let N be the set of nodes and E the set of arcs in G . We define the function r which assigns to each arc e in E an ordered pair of distinct nodes (j,k) , j and k in N , such that j is the beginning node of e and k its ending node. r induces in a natural fashion the function r_1 and r_2 defined as

$$r(e) = (j,k) = [r_1(e), r_2(e)] .$$

The nodes are labelled by $i (i = 1, \dots, n)$ in such a way that 1 is the origin node, n the objective node and if there is an arc from node j to node k then j is less than k . The arcs are labelled, in any order, by $e (e = 1, \dots, m)$. θ_i is the set of arcs e such that $r_2(e) \leq i$

$$\theta_i = \{e \text{ in } E \mid r_2(e) \leq i\} .$$

V_i is a subset of θ_i defined at the present stage as

$$V_i = \{e \text{ in } E \mid r_2(e) = i\} .$$

E_i and $\bar{E}_i$ are any two disjunctive subsets of θ_i such that $E_i \cup \bar{E}_i = \theta_i$.

With each arc e , we associate a random variable t_e which is the duration of the corresponding activity. The duration of the project up to node i is denoted ℓ_i . It follows that ℓ_1 equals 0 and that

ℓ_n is the total completion time of the project.

We will now state a result based on Jensen's inequality, Rao (1971), that will be useful thereafter:

Proposition 1

If $X_1, \dots, X_s$ are random variables then

$$E_{X_1, \dots, X_s} \left[\max_{j \in \{1, \dots, s\}} \{f_j(X_1, \dots, X_s)\} \right] \geq \max_{j \in \{1, \dots, s\}} \left\{ E_{X_1, \dots, X_s} [f_j(X_1, \dots, X_s)] \right\}$$

where $E_{X_1, \dots, X_s}$ is the mathematical expectation involving random variables

$X_1, \dots, X_s$ and f_j is a function defined on those random variables.

3. Lower bounds for the expected completion time

We are now concerned with lower bounds for the expected time required to complete all the activities of the project. Taking into account the definition of ℓ_i , we can write for $1 < i \leq n$,

$$\ell_i = \max_{v_i} \{ \ell_{r_1(e)} + t_e \} .$$

Taking the mathematical expectation on each side, we have

$$E[\ell_i] = E_{T_{\theta_i}} [\ell_i] = E_{T_{\theta_i}} [\max_{v_i} \{ \ell_{r_1(e)} + t_e \}]$$

where $T_{\theta} = \{t_e \mid e \text{ in } \theta\}$. If we divided the set T_{θ_i} in two parts,

we can write

$$E[\ell_i] = E_{T_{E_i}} E_{T_{\bar{E}_i}} [\max_{v_i} \{ \ell_{r_1(e)} + t_e \}]$$

where $E[\ell_i]$ is the expected completion time up to node i . Proposition 1 allows us to write the following inequality

$$E[\ell_i] > E_{T_{E_i}} [\max_{v_i} \{ E_{T_{\bar{E}_i}} [\ell_{r_1(e)} + t_e] \}] .$$

This inequality gives a general lower bound for $E[\ell_i]$.

Let us look at some particular configurations for E_i .

i) $E_i = \emptyset$ (the empty set)

If E_i is equal to the empty set, the bound reduces to

$$E[\ell_i] > \max_{V_i} \{ E_{T_{\theta_i}} [\ell_{r_1(e)}] + E_{T_{\theta_i}} [t_e] \} \text{ for } 1 < i \leq n.$$

If we define recursively

$$g_1 = 0$$

and

$$g_i = \max_{V_i} \{ g_{r_1(e)} + E_{t_e} [t_e] \} \text{ for } 1 < i \leq n,$$

g_i ($i = 1, \dots, n$) represents the well known bound obtained by the CPM.

It is easily shown that g_i is a lower bound for $E[\ell_i]$:

$$g_i \leq E[\ell_i] \text{ for } 1 \leq i \leq n.$$

Clearly this is true for $i = 1$, and we proceed by induction.

$$\text{ii) } E_i = V_i$$

For E_i defined as the set of incoming arcs at node i , the independence assumption allows to write the bound as

$$E[\ell_i] > E_{T_{V_i}} \{ \max_{V_i} \{ E_{T_{\bar{V}_i}} [\ell_{r_1(e)}] + t_e \} \} \text{ for } 1 < i \leq n$$

where $\bar{V}_i = \{e \text{ in } E \mid r_2(e) < i\}$.

If we define recursively

$$f_1 = 0$$

and

$$f_i = E_{T_{V_i}} \{ \max_{V_i} \{ f_{r_1(e)} + t_e \} \} \text{ for } 1 < i \leq n,$$

f_i ($i = 1, \dots, n$) represents the Fulkerson's estimate for the discrete and continuous cases. The improved bound proposed by Elmaghraby follows almost directly if we reverse the direction of the arcs e in θ_i . It is easily shown again by induction that

$$f_i \leq E[\ell_i] \quad \text{for } 1 \leq i \leq n.$$

We obviously have by proposition 1

$$f_i \geq \max_{v_i} \{f_{r_1}(e) + E[t_e]\}$$

and it follows that $g_i \leq f_i$ for $1 \leq i \leq n$ as expected.

$$\text{iii) } E_i = \theta_i$$

If E_i equals θ_i then the lower bound reaches the expected value:

$$E[\ell_i] \geq E_{T_{E_i}} \left[\max_{v_i} \{ \ell_{r_1}(e) + t_e \} \right] = E_{T_{\theta_i}} [\ell_i],$$

in general as stated before the evaluation of $E[\ell_i]$ involves a lot of work.

We can collect in E_i all arcs in θ_i that we judge relevant in so far as E_i permits to evaluate a satisfying bound in an efficient way. In the next section, we introduce a new bound where E_i meets the above requirements.

4. A new bound

We will generalize the Fulkerson's bound; to this end we introduce a few definitions. We denote p as a path in G . We extend the definition of r over P the set of all p .

$$r(p) = (j, k) = [r_1(p), r_2(p)] \quad p \text{ in } P$$

where j is the beginning node of p and k its ending node. With each p in P , we associate a random variable t_p which is the total duration of the corresponding set of activities associated with p . Let us define $d^+(j)$ the number of incoming arcs at node j and $d^-(j)$ the number of outgoing arcs at node j . For convenience, we will focus on node i , $1 < i \leq n$. Let P_i any subset of P such that $r_2(p) = i$. Taking into account the definition of ℓ_i , we can write

$$\ell_i = \max_{P_i} \{ \ell_{r_1(p)} + t_p \}.$$

From node i , instead of looking back to the nodes defined by $r_1(e)$ e in V_i , we go back up to the nodes $r_1(p)$ for p in P_i . This leads to the following form for a general lower bound for $E[\ell_i]$:

$$E[\ell_i] \geq E_{T_{E_i}} [\max_{P_i} \{ E_{T_{E_i}} [\ell_{r_1(p)} + t_p] \}].$$

In view to obtain a particular bound, we have to specify P_i and E_i : p defined by nodes $j = j_1, \dots, j_\ell = i$ is in P_i if

$$d^+(j_k) = d^-(j_k) = 1 \quad \text{for } k = 2, \dots, \ell - 1,$$

$$d^+(i) = 1$$

and

$$d^+(j) \neq 1.$$

All the nodes belonging to p in P_i have only one incoming arc and only one outgoing arc, except the nodes $r_1(p)$ which have zero or more than one incoming arcs. E_i is defined as follows

$$E_i = \{e \text{ in } \theta_i \mid e \text{ belongs to } p \text{ in } P_i\}.$$

Let us define recursively

$$m_1 = 0$$

and

$$m_i = E_{T_{E_i}} \left[\max_{P_i} \{m_{r_1(p)} + t_p\} \right] \quad \text{for } i < i \leq n.$$

It is a generalization of the Fulkerson's bounds. For a given m_i , we are faced with the evaluation of the expected time through a reduced PERT network having a particular structure: all the paths from the origin node to the objective node are parallel. Figure 1 illustrates such a parallel PERT network where the path duration random variables appear along the last arc of each path. s denotes the number of paths in P_i . $D_0, D_1, \dots$, and D_s are dummy nodes and $(D_0, D_j), 1 \leq j \leq s$, dummy arcs with constant associated duration equal to $m_{r_1(p_j)}$.

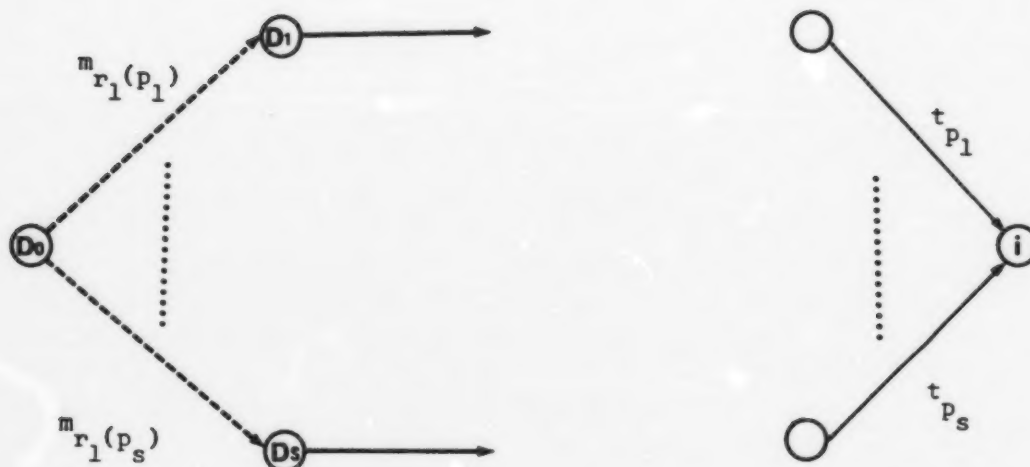


Figure 1. An s-parallel PERT network.

Let T the completion time through the parallel paths:

$$T = \max_{P_i} \{m_{r_1}(p) + t_p\} .$$

If the density function of T is easily accessible, we may obtain directly m_i . But this situation is unusual; however Martin (1965) presents a method for the efficient computation of the density function of the completion time in PERT networks under the restriction that the arc density functions are polynomials. This method is pertinent to our problem on account of the particular structure of the PERT network (see Figure 1). Indeed, according to Martin's notation, we obviate the trans-

formation which reduces a PERT network to a series-parallel network. Moreover, the reduction of the series-parallel network simply consists in transforming a parallel network containing s paths to a single equivalent arc with duration time T . When the arc distribution functions are polynomials the evaluation of m_i is relatively straightforward. The class of density functions that can be written as polynomials provides a rich selection of functions: uniform distribution, beta distribution with integer parameters. Obviously, we have

$$f_i \leq m_i \leq E[\ell_i] .$$

In general, it is difficult to evaluate m_i because we have to handle many t_p at the same time. It is possible to obtain a bound n_i satisfying

$$f_i \leq n_i \leq E[\ell_i]$$

where we have to deal only with the t_e .

We conclude this section by describing three possible definitions for n_i .

First, we may replace each duration random variable on the arc e in E_i by its expected value except for the arc e in E_i for which $r_2(e) = i$ and use the bound proposed by Fulkerson on the reduced PERT network. If we denote the resulting value n_i^1 , we have $f_i = n_i^1$.

The second approach is based on the following observation: in the reduced PERT network, if we permute the arcs along a path, the average duration of the partial project remains unchanged. However, if we calculate n_i^1 for various permutation, n_i^1 does not remain necessarily

the same. For the reduced network of Figure 2 where t_e assumes the values written on arc e with equal probabilities, n_i^1 equals 6.5. If we permute arcs (1,3) and (3,i), n_i^1 becomes 6.75 and if we permute arcs (1,3), (3,i) and arcs (1,2) (2,i) n_i^1 equals 6.16. We choose n_i^2 as the maximum value of n_i^1 over all the relevant permutations:

$$f_i < n_i^2 < m_i < [l_i] .$$

The third approach that produces n_i^3 , use the bound proposed by Elmaghraby (1967) based on the observation that we may reverse the direction of the arcs in a PERT network without changing the average duration of the project, but at each step we use n_i^2 instead of f_i . So

$$f_i < s_i < n_i^3 < E[l_i]$$

where s_i is the Elmaghraby initial bound. Note that n_i^3 may exceed m_i .

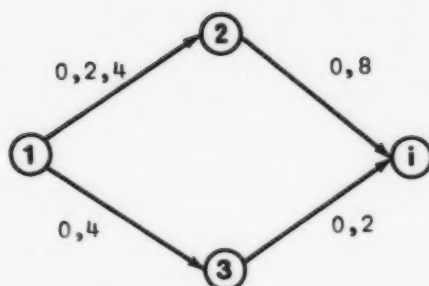


Figure 2. Example of a reduced PERT network.

Among all those lower bounds presented and those that follow in a natural fashion, we have to choose the one that satisfies our requirements in precision and in cost: as it is in almost all cases, the improved results is obtained at a certain cost, in the form of extra computing effort.

REFERENCES

- Burt, John, M. and Garman, Mark, B., "Monte Carlo Techniques for Stochastic PERT Network Analysis", Operations Research, Vol. 13, No. 1, 1965, pp. 46-66.
- Clingen, C.T., "A Modification of Fulkerson's Algorithm for Expected Duration of a PERT Project when Activities have Continuous d.f.", Operations Research, Vol. 12, No. 4, 1964, pp. 629-632.
- Elmaghraby, Salah, E., "On the Expected Duration of PERT Type Networks", Management Science, Vol. 13, No. 5, 1967, pp. 299-306.
- Fulkerson, D., R., "Expected Critical Path Lengths in PERT Networks", Operations Research, Vol. 10, No. 6, 1962, pp. 808-817.
- Kaufmann, A. and Desbazeille, G., "La méthode du chemin critique", Dunod 1969, 2e édition.
- Lindsey, J., H., II, "An Estimate of Expected Critical-Path Length in PERT Networks", Operations Research, Vol. 20, No. 4, 1972, pp. 800-812.
- MacCrimmon, K., R. and Ryavec, C., A. "An Analytical Study of the PERT Assumptions", Operations Research, Vol. 12, No. 1, 1964, pp. 16-37.
- Martin, J., J., "Distribution of Time Through a Directed, Acyclic Network", Operations Research, Vol. 13, No. 1, 1965, pp. 46-66.

Moder, Joseph, J. and Phillips, Cecil, R., "Project Management with CPM and PERT", Second Edition, Van Nostrand Reinhold Company, 1970.

Pritsker, Alan, B. and Kiviat, Philip, J., "Simulation with GASP II", Prentice-Hall, Inc., 1969.

Rao, Radhakrishna, C., "Linear Statistical Inference and its Applications", Second Edition, John Wiley and Sons, Inc., 1971.

Robillard, Pierre and Trahan, Michel, "The Completion Time of PERT Network", Publication No. 157, Département d'informatique, Université de Montréal, January 1974.